In this exercise we are considering a horizontal (along \( z \)), periodic (distance \( a \)), linear chain of identical pendulums (mass \( m \), moment of inertia \( I \), and distance \( L \) from the chain) coupled by cylindrical elastic rods (Lamé coefficients \( \mu \), \( \lambda \) and and radii \( R_1 \), \( R_2 \)) in the presence of gravity \( g \). The pendulums oscillate perpendicular to the \( z \) axis. The position of the \( n \)th pendulum is described by the angle \( u_n \) from vertical.

![Figure 1: The geometry of the considered problem.](image)

**1 Torsional Rigidity of the Elastic Rods**

If the rod is twisted the small angle \( \tau \) per unit length the resulting torque is \( M = C \tau \), where \( C \) is the torsional rigidity, which will be calculated in this section. For a given surface element \( dS \) we have \( dM = x \times \sigma \cdot dS \). Therefore we are interested in determining \( dF = \sigma \cdot dS \).

At position \( z \) the rod is rotated \( \tau z \) (pure torsion assumed), and the displacement on the outer border at this \( z \) is:

\[
\mathbf{u} = \tau z \hat{\mathbf{z}} \times \mathbf{x} = \tau z \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = \tau z \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad (1)
\]

From this result we get \( \nabla_i u_x = 0 \), \( \nabla_x u_x = 0 \), and \( \nabla_y u_y = 0 \) which means that the only gradients different from zero is \( \nabla_y u_x \), \( \nabla_z u_x \), \( \nabla_x u_y \), and \( \nabla_z u_y \). This is used to find the non-zero elements of Cauchy’s strain tensor:

\[
u_{ij} = \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) \quad (2)
\]
from where we immediately get the only existing strain elements in our problem:

$$\{u_{ij}\} = \begin{bmatrix} 0 & 0 & u_{xz} \\ 0 & 0 & u_{yz} \\ u_{xz} & u_{yz} & 0 \end{bmatrix}, \quad u_{xz} = -\frac{1}{2} \tau y, \quad u_{yz} = \frac{1}{2} \tau x$$ \quad (3)

The stresses can now be found with Hooke’s law:

$$\sigma_{ij} = 2\mu u_{ij} + \lambda \delta_{ij} \sum_k u_{kk}$$ \quad (4)

where the last term is the stress when a principal basis is used, and the first term is what is added if the coordinate system is not the principal basis. The existing stresses in the considered problem is:

$$\{\sigma_{ij}\} = \begin{bmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{bmatrix}, \quad \sigma_{xz} = -\mu \tau y, \quad \sigma_{yz} = \mu \tau x$$ \quad (5)

This result clearly satisfies Cauchy’s strain tensor and Hooke’s law since it is derived from these. That the result satisfies the third and last differential equation of elastostatics; mechanical equilibrium:

$$f_i + \sum_j \nabla_j \sigma_{ij} = 0$$ \quad (6)

is true since we have no volume forces and $$\nabla_x \sigma_{xx} = \nabla_z \sigma_{xz} = \nabla_y \sigma_{yz} = 0$$ and $$\nabla_y \sigma_{xz} + \nabla_x \sigma_{yz} = 0$$.

Now the necessary quantities needed for determining $$\text{d}F$$ is known:

$$\text{d}F = \sigma \cdot \text{d}S = \begin{bmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \text{d}S_z \end{bmatrix} = \begin{bmatrix} \sigma_{xz} \text{d}S_z \\ \sigma_{yz} \text{d}S_z \\ 0 \end{bmatrix}$$ \quad (7)

which is used to calculate the resulting torque on the surface element $$\text{d}S$$:

$$\text{d}M = \mathbf{x} \times \sigma \cdot \text{d}S = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \\ x & y & z \\ \sigma_{xz} \text{d}S_z & \sigma_{yz} \text{d}S_z & 0 \end{bmatrix} = \begin{bmatrix} -z\sigma_{yz} \text{d}S_z \\ z\sigma_{xz} \text{d}S_z \\ (x\sigma_{yz} - y\sigma_{xz}) \text{d}S_z \end{bmatrix}$$ \quad (8)

Since we only have the surface element $$\text{d}S_z$$ the torque only has a z component ($$\int_A f \text{d}S_{|i=x,y} = 0$$) – this also means that we can disregard/leave out the two first rows in the above matrix. The total torque in the z direction is now calculated:

$$M_z = \int_A (x\sigma_{yz} - y\sigma_{xz}) \, \text{d}S_z = \mu \tau \int_A (x^2 - y^2) \, \text{d}S_z$$

$$= \mu \tau \int_{R_1}^{R_2} \int_0^{2\pi} r^2 \, d\phi \, dr = 2\pi \mu \tau \left[ \frac{R^4}{4} \right]_{R_1}^{R_2}$$

$$= \frac{\pi \mu \tau}{2} (R_2^4 - R_1^4)$$ \quad (9)
Actually one could write up the integral without the preceding derivations since one from the cylindrical geometry would have guessed that we had to integrate over $r^2$. But by including the arguments wrong guesses can be avoided. The scope of this section is to find the torsional rigidity $C$ (units Nm$^2$) which can be stated from the above result:

$$C = \frac{M}{\tau} = \frac{\pi \mu}{2} (R_2^4 - R_1^4)$$ (10)

## 2 Equation of Motion and the Energy for the $n$th Section of the Chain

The equation of motion for the system (discrete sine-Gordon) is:

$$I \frac{d^2 u_n}{dt^2} = -Lmg \sin u_n + C \frac{u_{n-1} - u_n}{a} - C \frac{u_n - u_{n+1}}{a}$$

$$= -Lmg \sin u_n + \frac{C}{a} (u_{n-1} + u_{n-1} - 2u_n)$$ (11)

where (in the first line) the first term is the torque due to gravity on the pendulum, second term the torque due to the $\Delta u_n$ between the considered section and the section before, and the third term is due to the $\Delta u_n$ with the section after the considered one. The difference in angle is of course scaled with the section distance $a$ since the torque is given by $C \tau$, where $\tau$ is the twisted angle per unit length. One can convince oneself that the signs are correct, by considering each sign on the right hand side: (i) a pendulum will because of the gravitational force try reversing a positive rotation, (ii) a positive twist before the considered section will rotate the section in the positive direction, and (iii) if a positive twist is also applied with respect to the next section then this will cause a smaller effective twist.

The energy of the $n$th section on the chain is:

$$E_n = E_{\text{kin, pendulum}} + E_{\text{pot, pendulum}} + E_{\text{elastic}}$$ (12)

$$= \frac{1}{2} I \left( \frac{du_n}{dt} \right)^2 + mgL (1 - \cos u_n) + \frac{1}{2} \int_V \sum_{i,j} \sigma_{i,j} u_{i,j} dV$$

The elastic energy (for an isotropic material) is evaluated:

$$E_{\text{elastic}} = \frac{1}{2} \int_V \sum_{i,j} \sigma_{i,j} u_{i,j} dV = \frac{1}{2} \int_V (2\sigma_{x,z} u_{x,z} + 2\sigma_{y,z} u_{y,z}) dV$$

$$= \frac{1}{2} \int_V (\mu \tau^2 y^2 + 2\mu \tau^2 x^2) dV = \frac{1}{2} \tau^2 \mu \int_V (y^2 + x^2) dV$$

$$= \frac{1}{2} \tau^2 \mu \int_{R_1}^{R_2} \int_0^{2\pi} r^2 \tau d\phi dr dz' = \frac{1}{2} \tau^2 \int_z^{z+a} C dz'$$

$$= \frac{1}{2} \tau^2 a C = \frac{1}{2} \left( \frac{u_{n+1} - u_n}{a} \right)^2 a C = \frac{1}{2} C \left( \frac{u_{n+1} - u_n}{a} \right)^2$$ (13)

Thus the energy of the $n$th section becomes:

$$E_n = \frac{1}{2} I \left( \frac{du_n}{dt} \right)^2 + mgL (1 - \cos u_n) + \frac{1}{2} C \left( \frac{u_{n+1} - u_n}{a} \right)^2$$ (14)

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3 Going to the Continuum Limit

In this section the equation of motion and the total energy is obtained in the continuum limit because the discrete sine-Gordon system cannot be solved analytically. To assist us in the translation it proves helpful to look at the definition of the derivative (small $a$):

\[
\begin{align*}
  u_z &= \frac{u_{n+1} - u_n}{a} \quad (15) \\
  u_{zz} &= \frac{u_{z,n+1} - u_{z,n}}{a} = \frac{(u_{n+1} - u_n) - (u_n - u_{n-1})}{a^2} \\
  &= \frac{u_{n+1} + u_{n-1} - 2u_n}{a^2} \quad (16)
\end{align*}
\]

By using this the equation of motion (11) can now be transformed to the continuum limit:

\[
\begin{align*}
  I \phi_{tt} &= -Lmg \sin \phi + aC \phi_{zz} \quad \Leftrightarrow \quad aC \phi_{zz} - I \phi_{tt} = Lmg \sin \phi \\
  \phi_{zz} - \frac{1}{c^2} \phi_{tt} &= \frac{1}{\lambda^2} \sin \phi \\
  \phi_{zz} - \frac{1}{c^2} \phi_{tt} &= \frac{1}{\lambda^2} \sin \phi \\
  c &= \sqrt{\frac{aC}{I}} \quad \land \quad \lambda = \sqrt{\frac{aC}{Lmg}} \quad (17)
\end{align*}
\]

where $\lambda$ is called the characteristic length and $c$ the characteristic velocity.

The energy in the continuum limit is found by integrating over the energy density along the $z$ axis.

\[
E = \int \mathcal{E} \, dz \quad (18)
\]

The energy density is found by taking the energy (14) in the continuum limit and dividing by $a$:

\[
\begin{align*}
  \mathcal{E} &= \frac{1}{2} \frac{I}{a} \phi_t^2 + \frac{mgL}{a} (1 - \cos \phi) + \frac{1}{2} \frac{C}{a^2} (a \phi_z)^2 \\
  &= C \left( \frac{1}{2} \frac{I}{aC} \phi_t^2 + \frac{mgL}{aC} (1 - \cos \phi) + \frac{1}{2} \frac{C}{a^2} \phi_z^2 \right) \\
  &= C \left( \frac{1}{2} \frac{I}{c^2} \phi_t^2 + \frac{1}{\lambda^2} (1 - \cos \phi) + \frac{1}{2} \phi_z^2 \right) \quad (19)
\end{align*}
\]

*From now on indices $t$ and $z$ on a quantity means derivative with respect to the indices.*

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4 Stationary Solution to Continuum sine-Gordon and Moving 2π-kinks

In the stationary case ($\phi_{tt} = 0$) the continuum sine-Gordon system can easily be solved analytically. In the following the system is solved with the boundary condition $\phi(\pm \infty) - \phi(\mp \infty) = 2\pi$ (a $2\pi$ twist). To solve the system it is convenient to start out by multiplying with $\frac{d\phi}{dz}$ and using that $\frac{d}{dz} \left( \frac{d\phi}{dz} \right)^2 = 1$ 

\[ \frac{d^2 \phi}{dz^2} \frac{d\phi}{dz} = 1 \]

\[ \frac{d}{dz} \left( \frac{d\phi}{dz} \right)^2 = \frac{1}{\lambda^2} \sin \phi \frac{d\phi}{dz} \quad \Leftrightarrow \quad \frac{1}{2} \frac{d}{dz} \left( \frac{d\phi}{dz} \right)^2 = \frac{1}{\lambda^2} \sin \phi \frac{d\phi}{dz} \quad \Leftrightarrow \quad \left( \frac{d\phi}{dz} \right)^2 = \frac{2}{\lambda^2} \int_0^\phi \sin \phi' \, d\phi' = \frac{2}{\lambda^2} (1 - \cos \phi) + k(0) \quad (20) \]

where we must select the constant of integration to satisfy the boundary condition. Since these imply $\frac{d\phi}{dz} = 0$ we get $k(0) = 0$. Transforming this result to a one-term sine proves helpful later. To do this a relation is derived from the relation of double angles and the basic relation:

\[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta \quad \Leftrightarrow \quad 2 \sin^2 \theta = 1 - \cos 2\theta \quad \rightarrow \quad 2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta \quad (21) \]

With this we obtain the solution:

\[ \left( \frac{d\phi}{dz} \right)^2 = \frac{4}{\lambda^2} \sin^2 \frac{\phi}{2} \quad \Leftrightarrow \quad \phi = 4 \arctan \left( \exp \left( \pm \frac{z}{\lambda} \right) \right) \quad (22) \]

This solution is called a $2\pi$-kink or just a kink for $'+$', and an antikink for $'-'$. Plots of these can be found in figure 3(a).† Figure 2 shows more illustrative plots of the solution in a 3D coordinate system.

The solution for a kink moving with velocity $v$ and unchanged shape can be found by using $\phi(z,t) = \phi(z - vt) = \phi(u)$. This gives by using the chain rule:

\[ \frac{d^2 \phi}{dt^2} = \left( \frac{du}{dt} \right)^2 \frac{d^2 \phi}{du^2} = v^2 \phi''_u \quad , \quad \frac{d^2 \phi}{dz^2} = \left( \frac{d\phi}{dz} \right)^2 \frac{d^2 \phi}{du^2} = \phi''_u \quad (23) \]

†Using standard notation in these calculations are more intuitive.
‡The constants $C$, $c$, and $\lambda$ are always chosen to be 1 in this exercise since these only scale the plots in various ways. It corresponds to dimensionless units.
By using this we can now rewrite (17) to:

$$\begin{align*}
\phi_{zz} - \frac{1}{c^2} \phi_{tt} &= \frac{1}{\lambda^2} \sin \phi \\
\phi_{uu} - \frac{v^2}{c^2} \phi_{tt} &= \frac{1}{\lambda^2} \sin \phi \\
\phi_{uu} &= \frac{1}{\lambda'^2} \sin \phi \\
\lambda &= \lambda \sqrt{1 - \frac{v^2}{c^2}} 
\end{align*}$$

We immediately see that this is the same problem as the problem we have just solved. Therefore we find the solution by insertion in (22), plot shown in figure 3(a):

$$\phi(z, t) = 4 \arctan \left( \exp \left( \pm \frac{u}{\lambda'} \right) \right)$$

$$= 4 \arctan \left( \exp \left( \pm \frac{z - vt}{\lambda' \sqrt{1 - \frac{v^2}{c^2}}} \right) \right)$$

When one considers figure 2 it might at a first glance seems odd that the twist is not evenly distributed along the z axis. The reason why this is not the case is that the confined/localised twist is energetically more favourable. To proof this we must examine that energy distributions of the two cases. The energy density for a moving kink is (cf. (19)):

$$\mathcal{E} = C \left( \frac{1}{2} \frac{1}{c^2} \phi_t^2 + \frac{1}{\lambda^2} \left( 1 - \cos \phi \right) + \frac{1}{2} \phi_z^2 \right)$$

$$\alpha = \pm \frac{z - vt}{\lambda' \sqrt{1 - \frac{v^2}{c^2}}}, \quad \alpha_t = \mp \frac{v}{\lambda' \sqrt{1 - \frac{v^2}{c^2}}}, \quad \alpha_z = \pm \frac{1}{\lambda' \sqrt{1 - \frac{v^2}{c^2}}}$$

$$\phi_t = 4\alpha_t \frac{\exp \alpha}{1 + \exp^2 \alpha}$$

$$\phi_z = 4\alpha_z \frac{\exp \alpha}{1 + \exp^2 \alpha}$$
Figure 3: (a) $(\phi, z)$ plot of kink, kink with $v = 0.8$, antikink, and of an evenly distributed twist. It is seen that when a kink has a velocity it narrows down. (b) Energy distribution for a kink/antikink and the evenly distributed twist.

and the energy distribution for an evenly distributed twist is (a cutout of length $d$ is considered):

$$E_{\text{even twist}} = C \left( \frac{1}{2} C^2 \phi_t'^2 + \frac{1}{\lambda^2} \left( 1 - \cos \phi_t' \right) + \frac{1}{2} \phi_z'^2 \right)$$

(27)

$$\phi_t' = \frac{z - vt}{d} \pi \ , \ \phi_t' = -\frac{v}{d} \pi \ , \ \phi_z' = \frac{\pi}{d}$$

It is immediately seen – as expected – that the energy is independent of the time since this will only translate the energy distribution along $z$. The energy distribution for zero velocity is shown in figure 3(b), and from this it is seen that the integral of the kink along $z$ is far smaller than the integral of the evenly distributed twist. This shows that the localised twist is more favourable.

When velocity is added the kink and the even twist gains more energy. In figure 4(a) $\phi_t'^2$'s velocity dependence is plotted. Figure 4(b) shows the energy distribution when velocity is introduced. Note that the velocity for the kink and the even twist cannot be compared since they are not normalised in the same way. However one, can from the indications of the plot see that the kink will always be more energetically favourable than the evenly distributed twist.
(a) Plot of $\phi_z^2$ for different velocities. (b) Energy distribution. Moving kink and twist included. Note that the velocities for the kink and twist are not comparable as described in the text (different normalisation).

5 Numerical Analysis of the sine-Gordon System

From $\phi_z$ in (26) and the energy plots we see that the kink is a very robust/localised solution. In this section we are going to study solutions that do change shape as a function of the time. This analysis is carried out with the aid of the MATLAB ODE programs found in the appendix.

5.1 Collision: kink-antikink

Figure 5 shows a kink and an antikink created in the same system. Since a(n) (anti)kink is very localised the system does not change over time since the distance with $\phi_z \approx 0$ is long enough to prevent interaction. The plots shown in this section show $z \in [0; 150], \ldots$
Figure 6: \((\phi, z)\) plots for different times for the kink-antikink system when the kink and antikink is placed close to each other. Interaction do occur for a larger distance than the used. Actually I should have used a distance that allows \([-2\pi; 2\pi]\).

but actually in dimensionless coordinates one should consider the \(z\) cutout to be of length 30 since \(e11=30\) in the MATLAB simulation. The longer \(z\) cutout is only used numerically to make the resolution better. (I just forgot to alter the \(z\) axis on all the plots afterwards).

When the distance between the kink and antikink is reduced interaction starts. Figure 6 shows the kink-antikink system with a short distance between the two centres. In the figure the time evolution is indicated. It shows that changes in the beginning are taking place very slowly, and accelerated the smaller the \(\phi\)-top in the \((\phi, z)\) plot becomes. For the negative side the speed of geometric changes is slowing down again till an configuration equal to the opposite of the initial configuration is reached. After this the exact same evolution takes place in the reverse order till the original configuration is reached. The process iterates forever when no damping is included.

To illustrate the actual geometric evolution better a series of 3D plots are included in figure 7. From these it is seen that we actually have the pendulums oscillating due to the fact that the gravitational attraction on the pendulums in the kink-antikink interface becomes smaller than the force arising from the torsion of the elastic rod.

The result of this simulation can also be used to understand what happens if a kink and antikink is moving towards each other. From the plots it is immediately seen that the kink and anti-kink ”swallows” each other and is recreated on the other side. This means the kink and antikink are transparent with respect to each other – an event very similar to when you stand exactly in the middle between to loud speakers sending out the same signal.

This transparency can also be simulated by locating the kink and antikink far away from each other and give them a velocity in different directions \((v \text{ and } -v, \text{ respectively})\). From the previous section we retrieve an expression for how we can set the correct velocity starting condition (plot of the collision may be found in figure 8(a)):

\[
\phi_t = 4\alpha_t \frac{\exp \alpha}{1 + \exp^2 \alpha}, \quad \alpha = \pm \frac{z - vt}{\lambda \sqrt{1 - \frac{v^2}{c^2}}}, \quad \alpha_t = \pm \frac{v}{\lambda \sqrt{1 - \frac{v^2}{c^2}}}
\]

(28)
Figure 7: 3D plot at different times for a kink and an antikink located close to each other. After the shown evolution it continues from \((f)\) in reverse order till the configuration in \((a)\) is obtained. This is one period of oscillation, which is repeated infinitely if no damping is included.

5.2 Collision: kink-kink

The collision of two kinks and two antikinks is naturally equivalent, which means that we only have to study one of the cases. In this section the kink-kink system is examined. Again we do not notice any interaction before the distance become small enough. Figure 8(b) (still \(t=30\)) shows the interaction that occurs when the two kinks are close to each other. Here it is seen the the kinks repel each other. Figure 9 shows this interaction for the colliding kinks.

Again a series of 3D plots are included for completeness in figure 10. From these one see that the repulsion occurs because the twisted angle per unit length becomes
Figure 8: \((\phi, z)\) plots. (a) A kink and an antikink colliding. Notice that both (anti)kinks continue without obstacles as if the other one is transparent. (b) Different times for the kink-kink system.

too large. The total energy can be lowered by splitting the kink-kink structure into two kinks. The energy that is freed in this process is transformed to kinetic energy. In the colliding example the kinks will naturally have the same velocity (with opposite sign) before and after the collision.

It is important to note that one has to use free boundary conditions when solving this problem in MATLAB, since we do not have periodicity (cf. figure 8(b)).

5.3 Breather Solution

In this section the evolution of the following starting conditions are examined:

\[
\phi(z, 0) = A \arctan \left( \frac{\sqrt{1 - \omega^2}}{\omega} \frac{1}{\cosh \left( \sqrt{1 - \omega^2} z \right)} \right), \quad \phi_t(z, 0) = 0
\]  
(29)

Simulations with \(A \in [2; 10]\) show that localised periodic solutions only exist for \(A = 4\). Figure 11 show simulation results for \(A = 2\) and \(A = 8\). In the plots it is seen

Figure 9: \((\phi, z)\) plot for two colliding kinks (repelling). Dashed: states after collision.
Figure 10: 3D plot at different times for two kinks located close to each other. Better resolution (at least larger np) should have been used for graph (f).
that the initial "kink" divides into more "kinks" moving away from each other ("kink-kink repulsion"). The reason for the "kinks" to divide up is that the twist per unit length is not energetically favourable, which means that the system tries minimising the energy by changing its shape. That $A = 4$ gives localised solutions could have been guessed on beforehand since this was the $\arctan(...)$ coefficient used in the previous solutions – which had energetically favourable shapes.

The localised solutions one find are called breathers. Figure 12(a) shows that $\omega$ determines the size of the swing that the breather performs. In 12(b) the time evolution of a simulation with $\omega = 0.5$ is given. Figure 13 shows a 3D plot of the simulation. The results from this simulation look very similar to the kink-antikink simulation except that we do not have $2\pi$ twisting (anti)kinks. If one start looking for an analytical solution one will also find a solution similar to the one of an kink-antikink system.

Figure 11: $(\phi, z)$ plots at different times. (a) $A = 2$. (b) $A = 8$.

Figure 12: $(\phi, z)$ plots. (a) The role of $\omega$, $A = 4$. (b) Simulation result for a localised breather.
6 Equilibrium in a Rectangular Elastic Rod

Instead of the hollow circular elastic rod a solid elastic rectangular rod is now considered. The rod has thickness \( t \) and width \( w \). To fulfill the requirement that the tension in an area element must be zero (zero divergence) the displacement field should be replaced with:

\[
\mathbf{u} = \tau \begin{pmatrix}
-\frac{yz}{2} \\
\frac{xz}{2} \\
\psi(x, y)
\end{pmatrix}
\]  

(30)

where the torsion function \( \psi(x, y) \) of course must fulfill mechanical equilibrium and no surface tension on the rod. By repeating the procedure from section 1 we get:

\[
\{u_{ij}\} = \begin{bmatrix}
0 & 0 & u_{xz} \\
0 & 0 & u_{yz} \\
u_{xz} & u_{yz} & 0
\end{bmatrix}, \quad u_{xz} = \frac{1}{2} \tau (\psi_x - y), \quad u_{yz} = \frac{1}{2} \tau (\psi_y + x) 
\]

(31)

\[
\{\sigma_{ij}\} = \begin{bmatrix}
0 & 0 & \sigma_{xz} \\
0 & 0 & \sigma_{yz} \\
\sigma_{xz} & \sigma_{yz} & 0
\end{bmatrix}, \quad \sigma_{xz} = \mu \tau (\psi_x - y), \quad \sigma_{yz} = \mu \tau (\psi_y + x)
\]

(32)
Mechanical equilibrium sets constraints on $\psi(x, y)$. We have no volume forces, $f_i = 0$, which with the above result gives us (remember $\psi_{zz} = 0$):

$$0 = f_i + \sum_j \nabla_j \sigma_{ij} = \nabla_x \sigma_{xx} + \nabla_y \sigma_{zy} + \nabla_z \sigma_{xz} + \nabla_z \sigma_{yz} = \mu \tau \psi_{xx} + \mu \tau \psi_{yy} + 0 + 0 = \mu \tau \nabla^2 \psi \Rightarrow \nabla^2 \psi = 0 \quad (33)$$

Now we introduce the auxiliary function $\chi(x, y)$, which is determined by:

$$\chi_x = -\frac{1}{2} (x + \psi_y), \quad \chi_y = \frac{1}{2} (\psi_x - y) \quad (34)$$

Our stress components can be written as functions of these:

$$\sigma_{xz} = 2 \mu \tau \chi_y, \quad \sigma_{yz} = -2 \mu \tau \chi_x \quad (35)$$

That this definition of $\chi(x, y)$ fulfills mechanical equilibrium can easily be shown:

$$\nabla_x \sigma_{xx} + \nabla_y \sigma_{zy} = 2 \mu \tau (\chi_{yx} - \chi_{xy}) = 0 \quad (36)$$

where we have used that the order of differentiation does not matter. The definition of the function $\chi(x, y)$ that fulfills mechanical equilibrium also tells us that mechanical equilibrium requires:

$$\nabla^2 \chi = \chi_{xx} + \chi_{yy} = \nabla_x \chi_x + \nabla_y \chi_y = -\frac{1}{2} (1 + \psi_{yx}) + \frac{1}{2} (\psi_{xy} - 1) = -1 - \frac{1}{2} (\psi_{xy} - \psi_{yx}) = -1 \quad (37)$$

Now, all that there is left is to show what constraint we must put upon $\chi(x, y)$ to obtain zero surface tension (no forces acting on the surface). Zero surface tension requires $\sigma \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is a normal vector for the considered surface. By looking at the surfaces in the $x$ and $y$ direction we get (the same result is obtained by looking at the surfaces in the negative directions):

$$0 = \sigma \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \mu \tau \chi_y \end{pmatrix}$$

$$0 = \sigma \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \mu \tau \chi_x \end{pmatrix}$$

$$\Rightarrow \chi_y = 0 \quad \land \quad \chi_y = 0 \quad (38)$$

which means that we must require $\chi(x, y)$ to be constant along the surface to insure no surface tension.

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7 Torsional Rigidity of a Rectangular Elastic Rod

In this section we will develop an expression for torsional rigidity of the rectangular elastic rod. We set the origin of our \( x \) and \( y \) axes in the lower left corner of the rectangle. From section 1 we get:

\[
M_z = C\tau = \int_A (x\sigma_{yz} - y\sigma_{xz}) \, dS_z \\
= \int_A (-2x\mu\tau\chi_x - 2y\mu\tau\chi_y) \, dS_z \\
= \tau 4\mu \int_A \left(-\frac{1}{2}x\chi_x - \frac{1}{2}y\chi_y\right) \, dS_z \quad \Rightarrow \\
C = 4\mu \int_A \left(-\frac{1}{2}x\chi_x - \frac{1}{2}y\chi_y\right) \, dS_z 
\]

(39)

By using partial integration and remembering that \( \chi(x, y) = A \) on the surface we obtain for the first term:

\[
-\frac{1}{2} \int_0^t \int_0^w x\chi_x \, dx \, dy = -\frac{1}{2} \left(\int_0^t \int_0^w \chi \, dx \, dy - \int_0^t \int_0^w \chi_x \, dx \, dy\right) \\
= -\frac{1}{2} \left(\int_0^t \int_0^w \chi \, dx \, dy - \int_0^t \int_0^w \chi_x \, dx \, dy\right) \\
= -\frac{1}{2} \left((A - A)wt - \int_0^t \int_0^w \chi \, dx \, dy\right) \\
= \frac{1}{2} \int_0^t \int_0^w \chi \, dx \, dy 
\]

(40)

in the same manner we obtain the exact same result for the second term, and by insertion we get:

\[
C = 4\mu \int_0^t \int_0^w \chi(x, y) \, dx \, dy 
\]

(41)

The torsional rigidity \( C \) cannot be calculated analytically in the general case. But in the case of the infinitely wide rectangle (\( t \ll w \)) it can be found. We see that this corresponds to finding a flow profile \( \chi(x, y) \) in the infinitely wide rectangular pipe (IWRP) – by taking the surface constant \( A = 0 \) which corresponds to the no slip condition with the pipe wall. The flow profile in the IWRP is:

\[
\chi(x, y) = \begin{pmatrix} 0 \\ 0 \\ y(t - y)a \end{pmatrix} 
\]

(42)

where the constant \( a \) is determined by \( \nabla^2 \chi = -1 \) (\( \sim \)pressure gradient) to \( a = \frac{1}{2} \). The torsional rigidity for the wide rectangular rod can now be found (corresponds to the
volume flow): 

\[ C = 4\mu \int_0^t \int_0^w \frac{1}{2} y(t - y) \, dx \, dy \]
\[ = 2\mu w \int_0^t y(t - y) \, dy \]
\[ = \frac{1}{3} \mu wt^3 \]  \hfill (43)

If the rectangular rod is not infinitely wide the used \( \chi(x, y) \) does not fit the problem completely since it will not be constant in the ends of the \( y \) and \( -y \) directions. If the infinite assumption is not valid a more accurate analysis can be made by using Fourier series to represent \( \chi(x, y) \). Figure 14 plots the deviation between the flow profile in a \( w = 3t \) pipe and an infinitely wide rectangular pipe. Making such analyses for increasing \( w \) one obtain that the IWRP approximation is fairly good at a \( w = 30t \) geometry since the average error of the model is 1.75% in this case. For information about the origin of this number the reader is referred to the appendix.

\[ \text{Figure 14: Plot of the deviation between a flow profile} \ (\chi(x, y)) \text{ with} \ w = 3t \text{ and and infinitely wide rectangular pipe (rod).} \]
The material found in the appendix is not necessary for the reading of the assignment. It is only included to make the documentation more complete, and to allow easy reference for the author in the future.

A MATLAB Code

Used for Q4

The most important scripts used for this exercise.

% PHIOFZ.M
clf
lambda=1;
L=1;
figure(1)
hold on;
axis([-1 1 -1 1])
for z=-1:0.28:1
    phi=4*atan(exp(z/lambda))-pi/2;
    line([z z],[0 L*cos(phi)],[0 L*sin(phi)],'Color','b','LineWidth',0.6)
end;
line([-1 1],[0 0],[0 0],'Color','k','LineWidth',1.2)
xlabel('z');
ylabel('y=Lcos(\phi)');
zlabel('x=Lsin(\phi)');

% continuum plot
z=-1:0.05:1;
phi=4*atan(exp(z/lambda))-pi/2;
figure(2)
axis([-1 1 -1 1])
hold on;
for L=0:0.1:1
    plot3(z, L*cos(phi), L*sin(phi),'b-')
end;
xlabel('z');
ylabel('y=Lcos(\phi)');
zlabel('x=Lsin(\phi)');

% ENERGYDIST.M
lambda=1;
c=1;
v=0;
t=0;
d=10;
z=-1:0.01:1;
phi=4*atan(exp((z-v*t)/(lambda*sqrt(1-v^2/c^2)))); %phi for kink
v=0.8;
phi_v=4*atan(exp((z-v*t)/(lambda*sqrt(1-v^2/c^2)))); %phi for kink with velocity
v=0;
phi_anti=4*atan(exp(-(z-v*t)/(lambda*sqrt(1-v^2/c^2)))); %phi for antikink
phi_eve=phi_v+d/(d+pi); %phi for even twist distribution
figure(91)
plot(z,phi,'b-',z,phi_v,'b--',z,phi_anti,'b-.',z,phi_eve,'b:');
axis([-10 10 -0.217 6.5]);
xlabel('z');
ylabel('\phi');
legend('kink','kink (v=0.8)','antikink','even distributed twist');

% kink
pm=1; % or -1 for antikink
mp=-pm;
alpha=pm*(z-v*t)/(lambda*sqrt(1-v^2/c^2));
alpha_t=mp*v/(lambda*sqrt(1-v^2/c^2));
alpha_z=pm*1/(lambda*sqrt(1-v^2/c^2));
phi_t=4*alpha_t.*(exp(alpha))./(1+exp(alpha).^2);
phi_z=4*alpha_z.*(exp(alpha))./(1+exp(alpha).^2);
phi_t=4*alpha_t.*(exp(alpha))./(1+exp(alpha).^2);
E_kink = 1/2*phi_t.^2 + (1-cos(phi)) + 1/2*phi_z.^2;

% even distribution of twist
phi2_t=-v/(d)*pi;
phi2_z=pi/(d);
E_ed = 1/2*phi2_t^2 + (1-cos(phievendist)) + 1/2*phi2_z.^2;

figure(1)
plot(z,E_kink,'b-',z,E_ed,'b:');
xlabel('z');
ylabel('Energy distribution');
legend('kink','even distributed twist');

% examination of phi_t
phi_t_mat=zeros(1,length(z));
for i=0:1:9
    v=i/10;
    alpha=pm*(z-v*t)/(lambda*sqrt(1-v^2/c^2));
    alpha_t=mp*v/(lambda*sqrt(1-v^2/c^2));
    phi_t_mat(i+1,:)=4*alpha_t.*(exp(alpha))./(1+exp(alpha).^2);
    sum(phi_t_mat(i+1,:))
end;

figure(2)
plot(z,phi_t_mat(1,:).^2,'b',z,phi_t_mat(2,:).^2,'b',z,phi_t_mat(3,:).^2,'b',z,phi_t_mat(4,:).^2,'b',z,phi_t_mat(5,:).^2,'b',z,phi_t_mat(6,:).^2,'b',z,phi_t_mat(7,:).^2,'b',z,phi_t_mat(8,:).^2,'b',z,phi_t_mat(9,:).^2,'b');
xlabel('z');
ylabel('\phi_t^2');
axis([-10 10 0 8])

% kink (v=0.5)
v=0.5;

pm=1; % or -1 for antikink
mp=-pm;
alpha=pm*(z-v*t)/(lambda*sqrt(1-v^2/c^2));
alpha_t=mp*v/(lambda*sqrt(1-v^2/c^2));
alpha_z=pm*1/(lambda*sqrt(1-v^2/c^2));
phi_t=4*alpha_t.*(exp(alpha))./(1+exp(alpha).^2);
phi_z=4*alpha_z.*(exp(alpha))./(1+exp(alpha).^2);
phi_t=4*alpha_t.*(exp(alpha))./(1+exp(alpha).^2);
E_kink2 = 1/2*phi_t.^2 + (1-cos(phi)) + 1/2*phi_z.^2;

% kink (v=0.8)
v=0.8;

pm=1; % or -1 for antikink
mp=-pm;
alpha=pm*(z-v*t)/(lambda*sqrt(1-v^2/c^2));
alpha_t=mp*v/(lambda*sqrt(1-v^2/c^2));
alpha_z=pm*1/(lambda*sqrt(1-v^2/c^2));
phi_t=4*alpha_t.*(exp(alpha))./(1+exp(alpha).^2);
phi_z=4*alpha_z.*(exp(alpha))./(1+exp(alpha).^2);
phi_t=4*alpha_t.*(exp(alpha))./(1+exp(alpha).^2);
E_kink3 = 1/2*phi_t.^2 + (1-cos(phi)) + 1/2*phi_z.^2;

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Coupled Pendula – the sine-Gordon Equation
\% Even distribution of twist
\[ v = 0.8; \]
\[ \phi_{2 \text{e}} = \frac{v}{d \sqrt{1 - v^2/c^2}} \pi; \]
\[ \phi_{2 \text{t}} = \frac{v \pi}{d \sqrt{1 - v^2/c^2}}; \]
\[ \text{figure}(3); \]
\[ \text{plot}(x, \text{Eed}, 'r-', x, \text{Eed2}, 'r-.', x, \text{Ekink}, 'b-', x, \text{Ekink2}, 'b-.', x, \text{Ekink3}, 'b:'); \]
\[ \text{xlabel('\text{z}')}; \]
\[ \text{ylabel('Energy distribution')}; \]
\[ \text{legend('even twist (v=0)', 'even twist (v=0.8)', 'kink (v=0)', 'kink (v=0.5)', 'kink (v=0.8)'); \]
\[ \text{axis([-10 10 0 11])}; \]

A.1 Used for Q5

\% MAIN.M
\text{close all; clear workspace;}
\text{ell} = 30; \% domain length ... \text{dx}=1 for \text{ell}=\text{np}
\text{np} = 150; \% spatial grid
\text{tspan} = [0 10]; \% duration of simulation
\text{dx} = ell/np; \% spatial resolution
\% normalized, so in the following \text{c}=1, \text{lambda}=1

\text{distance} = 0.5 \% distance between placed (anti)kinks
\text{v} = 0.5
\text{omega} = 0.5
\text{A} = 4
\text{for} \text{i} = 1: \text{np}
\text{x} = i*\text{dx}-\text{ell}/2;
\% initial condition in phi
\text{u0}(\text{i}) = 4*atan(exp(x)); \% kink (v=0)
\text{u0}(\text{i}) = 4*atan(exp(-x)); \% antikink (v=0)
\text{u0}(\text{i}) = mod(u0(\text{i}),2*pi); \% u0 mod 2pi, used in kink-antikink (multipla solutions with 2pi are equal)
\text{u0}(\text{i}) = 4*atan(exp(x+distance)) + 4*atan(exp(-x-distance))); \% kink-antikink (v=0)
\text{u0}(\text{i}) = mod(u0(\text{i}),2*pi); \% u0 mod 2pi, used in kink-antikink (multipla solutions with 2pi are equal)
\text{u0}(\text{i}) = mod(u0(\text{i}),2*pi); \% u0 mod 2pi, used in kink-kink (multipla solutions with 2pi are equal)
\text{u0}(\text{i}) = A*atan(sqrt(1-omega^2)/omega * 1/(cosh(x*sqrt(1-omega^2)))); \% initial condition in velocity (\phi_t)
\text{u0}(\text{i+np}) = 0; \% zero velocity

\text{Solve for the given boundary conditions}
\text{[t, y]} = \text{ode45('waveq', tspan, u0, [], np, dx, ell1);}

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Coupled Pendula – the sine-Gordon Equation
function du=waveq(t,u,flag,np,dx,ell)
  %Boundary condition: 0=free, 1=periodic
  v=u(1:np); %phi
  vp=u(1+np:2*np); %dphi/dt

  %Create d^2phi/dz^2
  df=diag(-1/dx*ones(np,1))+diag(1/dx*ones(np-1,1),1); %forward difference matrix
  db=diag(1/dx*ones(np,1))+diag(-1/dx*ones(np-1,1),-1); %backwards diff
  % correction to the matrix in the upper left and lower right corner
  if bcon==1
    df(np,1)=1/dx;
    db(1,np)=-1/dx;
  end;
  dc=(db+df)/2; %central diff
  d2=dc*db; %2nd deriv.
  if bcon==0
    dc(1,1)=0.5/dx;
    dc(np,np)=0.5/dx;
    d2(np,np)=-1/(dx*dx);
  end;
  vxx=d2*v; %vxx=d^2phi/dz^2=dphi/dt

  % Equation to analyse: phi_zz - 1/c^2*phi_tt = 1/lambda^2*sin(phi)
  if c==1
    % <-> phi_tt = c^2*phi_zz - c^2/lambda^2*sin(phi)
    lambda=1;
    dv=vp; %dv=dphi/dt=phi_t
  end;
end;
\[ dp = c^2 \cdot v_{xx} - \frac{c^2}{\lambda^2} \sin(v) \] % \[ dp = c^2 \cdot \phi_{zz} - \frac{c^2}{\lambda^2} \sin(\phi) = \phi_{tt} \]

\[ du = [dv \quad \phi_t \quad \phi_{tt}] \] % \[ \phi_t \quad \phi_{tt} \]

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Coupled Pendula – the sine-Gordon Equation
B Part of an Exercise on Poiseuille Flow

Poiseuille Flow

33241 Theory of Lab-on-a-Chip Systems

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THIS IS PART OF AN EXERCISE MADE IN ANOTHER COURSE. IT IS ONLY INCLUDED TO SHOW HOW THE PLOT IN THE PRESENT ASSIGNMENT IS MADE AND WHERE THE DEVIATION NUMBER COMES FROM.

B.1 Poiseuille flow in a rectangular pipe – the structure of the velocity field

In a rectangular pipe, with the dimensions depicted in figure 15, there is a flow in the x direction with $-\Delta p$: $p(0) = p_0 + \Delta p$ and $p(L) = p_0$. Constant density $\rho(x, y, z, t) = \rho_0$ and the no-slip boundary condition are assumed fulfilled.

![Figure 15: Geometry of the rectangular pipe.](image)

Clearly the flow will only have a velocity field $U$ acting in the x direction. Furthermore $u_x$ can only be a function of the y and z since the system in translational invariant along the x axis. To summarise $U$ must be on the form:

$$U(x, y, z) = (u_x(y, z), 0, 0)$$ (44)

The continuity equation must of course be fulfilled to ensure mass conservation. This is checked:

$$\frac{\partial}{\partial t} \rho(x, y, z, t) = -\nabla \cdot J = -\nabla \cdot (\rho(x, y, z, t)U) \quad \land \quad \rho(x, y, z, t) = \rho_0$$

$$\Rightarrow \quad 0 = \nabla \cdot U$$

$$\nabla \cdot U = \partial_x u_x(y, z) + \partial_y 0 + \partial_z 0 \quad \text{where} \quad \partial_i \equiv \frac{\partial}{\partial t}$$

$$= 0$$ (45)

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B.2 Poiseuille flow in a rectangular pipe – Fourier transformation

With \( \rho = \rho_0 \), no acting body forces, and the structure (44) of \( \mathbf{U} \) the Navier-Stokes equation reduces to:

\[
\rho \frac{D}{Dt} \mathbf{U} = \rho \left( \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \mathbf{U} = -\nabla P + \eta \nabla^2 \mathbf{U} + \mathbf{f}
\]

\[\Rightarrow \nabla^2 u_x(y, z) = \frac{\nabla P}{\eta} = \frac{p(L) - p(0)}{\eta L} = -\Delta p \frac{\eta L}{\eta L} \quad (46)\]

The traditional time domain Fourier series is given by:

\[
f(t) \simeq \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(n\omega t) + b_n \sin(n\omega t) \right)\]

\[
a_n = \frac{2}{T} \int_{0}^{T} f(t) \cos(n\omega t) dt
\]

\[
b_n = \frac{2}{T} \int_{0}^{T} f(t) \sin(n\omega t) dt \quad (47)
\]

This series is easily transformed to the space domain by the substitution \( n\omega t = n \frac{\pi}{w} y \) or \( n\omega t = n \frac{\pi}{h} z \). The two-dimensional space domain Fourier series of \( u_x(y, z) \) is:

\[
u_x(y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin \left( n \frac{\pi}{w} y \right) \sin \left( m \frac{\pi}{h} z \right) \quad (48)
\]

where \( u_{nm} \) correlates the \( y \) and \( z \) dependence, so that in general \( u_x(y, z) \neq u_x(y)u_x(z) \). That \( u_x(y, z) \) is only given by sine functions is due to the boundary conditions. That (48) is a solution can be shown to be true by showing that it obeys the boundary conditions, and finding \( u_{nm} \) so that (46) is fulfilled. The boundary conditions are obeyed since:

\[
u_x(0, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin(0) \sin \left( m \frac{\pi}{h} z \right) = 0
\]

\[
u_x(w, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin(n\pi) \sin \left( m \frac{\pi}{h} z \right) = 0
\]

\[
u_x(y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin \left( n \frac{\pi}{w} y \right) \sin(0) = 0
\]

\[
u_x(y, h) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin \left( n \frac{\pi}{w} y \right) \sin(m\pi) = 0
\]

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One way of determining \( u_{nm} \) is by making an appropriate Fourier series of the right hand side of (46) and compare it to the left. The Fourier series of 1 is:

\[
\begin{align*}
& a_n = 2 \int_0^w \cos \left( \frac{n \pi}{w} y \right) \, dt = \frac{2}{n \pi} \left[ \sin \left( \frac{n \pi}{w} y \right) \right]_0^w = 0 \\
& b_n = 2 \int_0^w \sin \left( \frac{n \pi}{w} y \right) \, dt = \frac{2}{n \pi} \left[ \cos \left( \frac{n \pi}{w} y \right) \right]_0^w = \begin{cases} 
0 & \text{for } n \text{ even} \\
\frac{4}{n \pi} & \text{for } n \text{ odd}
\end{cases}
\end{align*}
\]

\Rightarrow 1 = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{1}{n} \sin \left( \frac{n \pi}{w} y \right) \tag{49}

Naturally, 1 can also be expressed in an equivalent form in the \( z \) direction. This implies that both sides of (46) take on similar forms:

\[
\begin{align*}
-\frac{\Delta p}{\eta L} : 1 \cdot 1 &= -\frac{16}{\pi^2} \frac{\Delta p}{\eta L} \sum_{n \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \frac{1}{nm} \sin \left( \frac{n \pi}{w} y \right) \sin \left( \frac{m \pi}{h} z \right) \\
\nabla^2 u_x(y, z) &= -\frac{\pi^2}{\eta L} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \left( \frac{n^2}{w^2} + \frac{m^2}{h^2} \right) \sin \left( \frac{n \pi}{w} y \right) \sin \left( \frac{m \pi}{h} z \right)
\end{align*}
\]

From this \( u_{nm} \) can be deduced, and it can be seen that only odd terms in (48) should be included:

\[
\begin{align*}
u_{nm} &= \frac{16}{\pi^4} \frac{\Delta p}{\eta L} \frac{1}{nm} \left( \frac{1}{w^2} + \frac{m^2}{h^2} \right) \\
u_x(y, z) &= \frac{16}{\pi^4} \eta L \sum_{n \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \frac{1}{nm} \left( \frac{1}{w^2} + \frac{m^2}{h^2} \right) \sin \left( \frac{n \pi}{w} y \right) \sin \left( \frac{m \pi}{h} z \right) \tag{51}
\end{align*}
\]

### B.3 Poiseuille flow in a rectangular pipe – the velocity field

A plot of the solution (51) of the velocity field is wanted. To make this one have to find \( n, m \) values that are large enough for the series to reach the limit of convergence. Figure 16 shows test plots that can be used to determine the needed values for \( n \) and \( m \). From these it is easily seen that the needed values grows rapidly for increasing \( h : w \).

Naturally the \( n/m \) value affect only the resolution along the \( y/z \) direction, which means that one can minimise the number of calculations needed to compute (51); for instance \( n \geq 100 \) is needed for \( h : w = 1 : 50 \) (figure 16.b) while only \( m = 35 \) is sufficient (figure 16.b). For simplicity \( n = m \) will be used in the plots.

The plots in this section have been made with the aid of the MATLAB code in section B.4. Unless otherwise stated the amplitude of all plots is normalised to 1.

Contour and surface plots for \( h : w = 1 : 1, 1 : 3, 1 : 10 \) are shown in figure 17 to 19.
Figure 16: Flow profile $u_x(y, h/2)$ for $h : w$ dimensions. For each plot the value of $n = m$ is shown.

Figure 17: Flow profile $u_x(y, z)$ for $h : w = 1 : 1$ dimensions. $n = m = 35$ is used. If higher resolution were employed a circular inner structure would be seen.

Figure 18: Flow profile $u_x(y, z)$ for $h : w = 1 : 3$ dimensions. $n = m = 35$ is used. A more elliptic inner structure would be seen with increased resolution.
Infinitely Wide Rectangular Pipe/Infinite Parallel Planes

The flow in the rectangular pipe can as seen in the plots be approximated with an infinitely wide rectangular pipe ($w \to \infty$) (IWRP) for increasing $h : w$. This can come in quite handy since computation of (51) can be quite time consuming. Following arguments similar to the ones used in section 3.1 one find the structure $U(x, y, z) = (u_x(z), 0, 0)$. Checking that the continuity equation is obeyed, and that Navier-Stokes reduces to (46) is trivial. By checking the no-slip boundary condition, and inserting in (46) it can be shown that the velocity profile for the IWRP with $h = 2a$ is:

$$u_{x}^{IWRP}(z) = z(2a - z) \frac{\Delta p}{2\eta L}$$

(52)

This profile is plotted in figure 20.

How large $h : w$ relationship that is required to obtain a good approximation with the IWRP is addressed in the following. From figures 18, 19, 21, and 22 one can see that the main error along the $y$ axis confines to $[0; a]$ and $[w - a; w]$ quite fast for increasing $h : w$. This is easily understood from the $1 : 1$ case from where it is obvious.
Table 1: Results from the error analysis. The mean value of \( u_{x}^{IWRP} \) is \( \langle u_{x}^{IWRP} \rangle = 7.5 \). Units of \( \Delta p / \eta L \).

<table>
<thead>
<tr>
<th>( (h+1)(w+1) \sum_{z=0}^{w} \sum_{z=0}^{h} \sigma_{yz} )</th>
<th>( h : w=1:3 )</th>
<th>( h : w=1:10 )</th>
<th>( h : w=1:30 )</th>
<th>( h : w=1:50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>max(( u_{x}^{finite} ))</td>
<td>12.267</td>
<td>12.515</td>
<td>12.508</td>
<td>12.548</td>
</tr>
<tr>
<td>max(( u_{x}^{IWRP} ))</td>
<td>12.5</td>
<td>12.5</td>
<td>12.5</td>
<td>12.5</td>
</tr>
</tbody>
</table>

that the wall affects the water to the a "depth". As an example the error for a 1 : 30 geometry approximated with the IWRP is less than \( 1/30 = 3.3\% \).

A better error analysis can be performed by looking at the deviations between the actual flows and the flow in the IWRP. Since these calculations involves only one point from each flow the deviation reduces to:

\[
\sigma = \sqrt{\mathcal{N}} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \mu)^2} \\
\mu = \frac{1}{N} \sum_{i=1}^{N} x_i \\
= \ldots = \frac{1}{\sqrt{2}} |x_1 - x_2| \tag{53}
\]

Figure 21 and 22 shows the results of this calculation on different geometries. If one sum up the errors and normalise by the geometry \( (h+1)(w+1) \) one find a number for how good the approximation is in each case; actually you get the average deviation in units of \( \Delta p / \eta L \) since this factor is omitted in the calculation of both. The results of such summations are listed in table 1 where the mean value of \( u_{x}^{IWRP} \) is also stated. With this method one find an average error of each coordinate on 0.13148/7.5 = 1.75% when the 1 : 30 flow is compared to the IWRP flow.

The values of max(\( u_{x}^{finite} \)) in table 1 might seem a bit odd since they exceed max(\( u_{x}^{IWRP} \)) = 12.5. The higher velocities comes from the ripple from the finite

\footnote{That 1 has to be added to each axis is due to the fact that a matrix containing \( u_{x} \) has rows and columns proportional to \( (w+1) \) and \( (h+1) \).}

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approximation. This ripple can be removed by choosing larger values for \( n \) and \( m \) (e.g., try comparing the values for 1 : 30 and 1 : 50).

By plotting the average deviations (\( \text{stdavg} \)) from table 1 (figure 23.a) it is seen that there is a potential dependence of \( \frac{w}{h} \). This is confirmed by the linear dependence one find by making a ln-ln plot (figure 23.b). An analytical expression for the deviation can easily be found:

\[
\ln(\text{stdavg}) = a \ln\left(\frac{w}{h}\right) + b
\]

\[\iff\] \[\text{stdavg} = \left(\frac{w}{h}\right)^a \cdot e^b\] (54)

By making the best linear fit \( a = -0.96916 \) and \( b = 1.2911 \) is found. A plot of the model is found in figure 23.

![Figure 23: Average deviations as a function of \( \frac{w}{h} \). Triangles show calculated points, and the curve shows the fitted model.](image)

**B.4 MATLAB Code**

The MATLAB code listed in this section is not prepared for presentation! It is ugly and only intended to help the author remember what commands that was used to make the calculations in MATLAB for this assignment.

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% Exercise 3
clear workspace;

% 1D plots for z=h/2 along y (actually it's the other way around, bet cuz of sym)

w = 10;
h = 10;
N = 1;
M = N;

u = zeros(w+1,h+1); %for 2D
%u = zeros(1,h+1); %for 1D

for y=0:w %for 2D
%y=w/2; %for 1D

for z=0:h

for n=1:N
for m=1:M
if (mod(n,2)+mod(m,2)==2) % only odd n and m's
u(y+1,z+1) = u(y+1,z+1) + 1/(n*m*(n^2/w^2 + m^2/h^2))*sin(n*pi*y/w)*sin(m*pi*z/h); %for 2D
%u(1,z+1) = u(1,z+1) + 1/(n*m*(n^2/w^2 + m^2/h^2))*sin(n*pi*y/w)*sin(m*pi*z/h); %for 1D
end
end
end

end %for 2D
u = u';
u = u/max(max(u)); % normalise

% 1D plot
% Make multiple plots in same figure by running the script multiple times with varying m and n
figure(10);
%plot([0:0.1:w/10, 0:0.1:h/10],u);
%ylabel('norm(u(y,h/2))');
%xlabel('y');
%hold on;

% 2D plots
plane = zeros(h+1,w+1);%for 2D

for z=0:h
plane(z+1,1) = z*(2*a-z);
end
for i=1:w
plane(:,i+1) = plane(:,1);
end

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u = u/max(max(u));  % normalise

% 2D plots
figure(1);
contourf(0:100:w*100, 0:0.1/5:z/50, u, 1000);
shading flat;
%figure(3)
%contour3(0:100:w*100, 0:0.1/5:z/50, u, 5000);

% Exercise 3 - Calc of avg std
clear workspace;
format long;
w = 300;
h = 10;
N = 120;
M = N;
u = zeros(w+1,h+1);  %for 2D
for y=0:w  %for 2D
    for z=0:h
        for n=1:N
            for m=1:M
                if (mod(n,2)+mod(m,2)==2)  % only odd n and m's
                    u(y+1,z+1) = u(y+1,z+1) + 1/(n*m*(n^2/w^2 + m^2/h^2))*sin(n*pi*y/w)*sin(m*pi*z/h);  %for 2D
                end
            end
        end
    end
end  %for 2D
u = u';
max(max(u))

u = u*16/pi^4;  %get in terms of Dp/(etaL)

% Infity rect pipe
a = h/2;
u2 = zeros(h+1,w+1);  %for 2D
for z=0:h
    u2(z+1,1) = z*(2*a-z);
end
for i=1:w
    u2(:,i+1) = u2(:,1);
end
max(max(u2))
u2 = u2/2;  %get in terms of Dp/(etaL)
error = 1/sqrt(2)*abs(u-u2);

% Exercise 3
% Plots for the error analysis
clear workspace;
errorX = [3 10 30 50];
errorY = [1.25949932267368 0.39042785855980 0.13148055121231 0.08367385765654];
figure(1);
plot(log(errorX),log(errorY),'k^');
hold on;
c = polyfit(log(errorX),log(errorY),1);
xpoints = [0 5];  %x points for the line
ypoints = polyval(c, xpoints);
plot(xpoints, ypoints, 'b-');

figure(2); %points in normal plot plotted along with model
x = 0:0.1:55;
y = x.^c(1) * exp(c(2));
plot(errorX, errorY, 'k^', x, y, 'b-');
axis([0 55 0 2]);